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Oscillatory disturbances as admissible solutions for the flow past a cylinder and their possible relationship to the Von Karman street phenomenon

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Summary

Evidently Proudman and Pearson's [1] low-Reynolds-number approximation scheme, as recently amended by the author, admits timewise oscillatory solutions. This is shown by relinquishing the implicit assumption that the flow is steady throughout, and letting instead each term in the asymptotic expansions consist of a sum of a steady component and a time-dependent one. When only one term is retained in the inner expansion, two terms in the outer and two in the recently-developed wake expansion, the solutions for the steady components are found to be determinable and equal to those recorded. However, the scheme also admits a large variety of non-trivial solutions for the time-dependent components. Attention is focused on those representing oscillatory modes of disturbance flow of indeterminable frequency and amplitude.

In the inner field such single mode has the form of a rotationally symmetric pattern. Far downstream it is in the form of a sequence of vorticity packets of alternate signs, equally spaced along the wake's centre plane. This pattern moves with the velocity of the undisturbed stream. The flow field resulting from such a disturbance superposed on a uniform stream bears a remarkable resemblance to the Von Karman vortex-street.

1. Introduction

The incompressible flow past a cylinder placed in a steady stream is governed by the Navier-Stokes equations together with boundary conditions which are imposed on the obstacle and at infinity. The former set of conditions reflects the requirement that all velocity components should vanish. At infinity one imposes the requirement that the disturbance created by the cylinder should be finite. The point made here is that although this differential system reflects a steady set-up, it admits solutions which represent timewise-oscillatory motion.

These admissible solutions are developed within the framework of Proudman and Pearson's [1] low-Reynolds-number matched-asymptotic-expansion scheme, as recently amended by the author [2]. It is thus assumed that, apart from an inner and an outer region, there is also a wake region which possesses distinct characteristics. Therefore, the unsteady solutions developed here, like the steady one presented in [2], consist of three expansions.

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These solutions are obtained as follows. The stream function in the inner, outer and wake regions is expressed by the small-Reynolds-number expansions recorded in [1] and [2]. Also, the scaling of the length coordinates in the three domains is adhered to. However, the proposed analysis is made more general by assuming that every term in each of the three expansions consists of a sum of a stationary and a time-dependent component. The characteristic time for the unsteady components of the inner expansions is a^2/ν. In the outer field and in the wake ν/U^2 is the characteristic time. Here a is the radius of the cylinder, ν is the kinematic viscosity and U is the velocity of the undisturbed stream.

This procedure is carried out so as to obtain one term in the inner expansion and two terms in each of the other two expansions. Obviously, the well-known solutions for the steady components are recovered. The time-dependent additions to the inner, the outer and the wake expansions are found to be governed, respectively, by the unsteady Stokes equation, the unsteady Oseen equation and a fourth-order equation in the co-ordinate normal to the flow direction. The former two were derived in Bentwich and Miloh's [3,4,5] analyses of the flow caused by a cylinder and a sphere departing impulsively from rest and acquiring a constant velocity. In these flows the unsteadiness soon dies out. In the flows under consideration, the uniform steady streaming motion is similarly dominant, whence the applicability of these equations to the case at hand.

Despite this mathematical similarity, Bentwich and Miloh's works on transient low-Reynolds-number flows radically differ from the present one. There, solutions were obtained for flow fields which meet all the requirements of certain differential systems. However, this is an investigation as to whether time dependence can emerge despite the fact that it is not induced. Evidently, there is such a possibility. Indeed, the procedure described admits a variety of time-dependent solutions. Many of these are of little interest because they represent unsteady disturbances in a highly-viscous flow, which decay and vanish either as time elapses or as they get convected and diffused throughout the infinitely wide stream. Attention is focused on disturbance-flow solutions which persist and have non-vanishing trace in all three domains, and it is found that timewise-oscillatory ones have these features. Within the framework of approximation adopted, the amplitude and frequency characterizing these cannot be determined. The net result is that in addition to the recorded steady solution for the flow under discussion, one gets persistent time-dependent indeterminable disturbances.

It follows that within the context of Proudman and Pearson's scheme the solution for the flow past a cylinder is not unique, unless steadiness is postulated. Indeed the author is unaware of a proof of uniqueness for unsteady two-dimensional flow past a cylinder which satisfies the requirement that the disturbance flow at infinity should be finite. (See the comprehensive and up-to-date study by Ladyzhenskaya [7]).

The article is concluded with a discussion of the refinements that could be achieved by calculating higher-order terms. In these subsequent stages of the analysis there is evidently coupling between the oscillatory and the steady components. It follows that the time-independent components of the higher-order terms in the asymptotic expansions are also indeterminable. So are the higher-order coefficients in the formula for the drag.

2. Analysis

The differential system governing the flow under discussion comprises the Navier-Stokes equations, together with conditions imposed on the velocity components on the cylindrical surface and at infinity. As explained, close to the cylinder, a and a^2/ν characterize space and time variations, respectively. The stream function is normalized with respect to Ua. Hence, in the inner field the governing equation reads

$$\frac{\partial}{\partial t}(\nabla^2\psi) + \mathrm{Re}\frac{\partial(\nabla^2\psi, \psi)}{\partial(x, y)} = \nabla^4\psi \tag{1}$$

where Re, the Reynolds number, is $Ua\nu^{-1}$. On the cylinder the following conditions prevail:

$$\psi = \partial\psi/\partial r = 0 \quad \text{at } r = 1. \tag{2),(3}$$

Here (x, y) are cartesian coordinates and the origin is fixed at the centre of the cylinder. They are related to the polar ones (r, θ) in the usual manner, and ∇^2 is the Laplace operator in these coordinates.

It is further assumed that in the outer field, that is, far from the obstacle and away from the half-plane downstream of the cylinder's axis, the characteristic length and time scales are independent of a. The first is the well-known viscous length (ν/U). The time scale appropriate for the outer field is ν/U^2, (see [3] and [5]). The independent variables characterizing variations in the outer field are therefore related to (r, t) by

$$R = r\mathrm{Re},\ T = t\mathrm{Re}^2.$$

It follows that the governing equation prevailing there reads

$$\frac{\partial}{\partial T}(\overline{\nabla}^2\psi) + \mathrm{Re}\frac{\partial(\overline{\nabla}^2\psi, \psi)}{\partial(X, Y)} = \overline{\nabla}^4\psi \tag{1'}$$

where (X, Y) are the outer cartesian coordinates, and $\overline{\nabla}^2$ is the Laplace operator in terms of these coordinates. At infinity the following conditions must be satisfied:

$$\psi \sim Y/\mathrm{Re} \quad \text{as } R \to \infty. \tag{4}$$

As shown in [2], the width of the wake is of $O(a)$, and variations across it are characterized by that length. The axial and the time variations there are as in the outer field. With these scaling the governing equation reduces to

$$\partial^4\psi/\partial y^4 + O(\mathrm{Re}^2) = 0 \tag{1''}$$

where the terms of $O(\mathrm{Re}^2)$ are negligible in the context of this work. It follows that the equation governing the stream function in the wake region does not contain X or T derivatives. Consequently the X and T variations inside the wake are those prevailing at its outer limit $y \to \infty$.

As explained, the non-uniqueness considered here is within the framework of Proudman and Pearson's structure of solution as amended by the author. Thus, there is an inner, an outer and a wake expansion which are expressed as follows:

$$\psi^{(i)} \sim \Delta\psi_1^{(i)}, \tag{5}$$

$$\psi^{(o)} \sim Y/\mathrm{Re} + (\Delta/\mathrm{Re})\psi_2^{(o)}, \tag{6}$$

$$(\partial\psi^{(w)}/\partial y,\ -\partial\psi^{(w)}/\partial x) = (1, 0) + O(\Delta). \tag{7}$$

The bracketed superscripts indicate regions, while the terms in each expansion are numbered sequentially. The gauge function Δ is defined as

$$\Delta = (\ln(1/\mathrm{Re}) + k)^{-1}.$$

Here k is a constant, and Kaplun [8] showed that there is merit in letting it be 3.703. The advantage of expressing the wake-flow solution in terms of the velocity components will become apparent later.

If steadiness is assumed to prevail throughout, then the leading inner term in Proudman and Pearson's solution is

$$\psi_1^{(i)} = \psi_{1s}^{(i)} = (r \ln r - r/2 + 1/2r) \sin\theta. \tag{8}$$

The subscript s is introduced here so as to stress that it was derived under that assumption. Moreover, to within errors of $O(\Delta/\mathrm{Re})$ and $O(\Delta)$, respectively, this matches expansions (6) and (7) when these are also assumed to be time-dependent. However, evidently the expression given by equation (8) together with the associated terms in the outer and wake expansions do not constitute the only admissible solution. The form given by relationships (5), (6) and (7) admits also time-dependent ones. They are derived by formally expressing the terms in these expansions as follows,

$$\psi_n^{(\;)} = \psi_{ns}^{(\;)} + \psi_{nu}^{(\;)}, \tag{9}$$

in which the bracketed superscript can be i, o or w and the subscript u implies unsteadiness. Since the flow at infinity is steady, it is required that $\partial\psi_{1u}^{(o)}/\partial y$ and $\partial\psi_{1u}^{(w)}/\partial y$ should vanish. However, there is no need to rule out *a priori* any of the other unsteady components.

The time-dependent components of the leading inner term is found to be governed by the unsteady Stokesian equation

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2\psi_{1u}^{(i)} = 0. \tag{10}$$

This is derived by substituting expansion (5) in equation (1) and retaining only the highest-order contribution. By similarly processing expansion (6), together with equation (1′), one finds that the time-dependent component of the disturbance stream function is governed by

$$\left(\frac{\partial}{\partial T} + \frac{\partial}{\partial X}\right)\overline{\nabla}^2\psi_{2u}^{(o)} = \overline{\nabla}^4\psi_{2u}^{(o)}, \tag{11}$$

which is the unsteady Oseen equation used in [3]. A time-dependent addition to the recorded steady-flow solution is consequently admissible if there are non-trivial solutions for $\psi_{1u}^{(i)}$, $\psi_{2u}^{(o)}$ and the corresponding time-dependent terms in expansion (7). These must satisfy equations (10), (11) and (1″), the matching requirements, as well as conditions (2), (3) and (4).

3. Admissible oscillatory modes

The following solution for $\psi_{lu}^{(i)}$ is proposed:

$$\psi_{1u}^{(i)}(r, \theta, t) = \int_{-\infty}^{\infty} \exp(\mathrm{i}\omega t)\Bigg\{A_0(\omega)\left[\frac{1}{\sqrt{\mathrm{i}\omega}}\left(K_0(\sqrt{\mathrm{i}\omega}\, r) - K_0(\sqrt{\mathrm{i}\omega})\right) + K_1(\sqrt{\mathrm{i}\omega}) \ln r\right]$$

$$+ \sum_{n=1}^{\infty} \begin{Bmatrix} A_n(\omega)\cos(n\theta) \\ B_n(\omega)\sin(n\theta) \end{Bmatrix}$$

$$\times\left[\frac{2}{\sqrt{\mathrm{i}\omega}} K_n(\sqrt{\mathrm{i}\omega}\, r) - \frac{1}{n} K_{n+1}(\sqrt{\mathrm{i}\omega})\frac{1}{r^n} + \frac{1}{n} K_{n-1}(\sqrt{\mathrm{i}\omega}) r^n\right]\Bigg\}\, \mathrm{d}\omega. \tag{12}$$

This is, in effect, a double summation of modes, each characterized by a frequency ω and a certain angular dependence. It is therefore a very general expression for the inner-field disturbance. Indeed, in terms of the Fourier integral with respect to ω and the summation over n, varieties of time and angular dependences, respectively, can be expressed. However, as shown below, this form gets considerably trimmed. So is the choice of mathematically admissable disturbances which are also of interest to this piece of research.

Examine first the admissibility of a single rotationally-symmetric mode designated by $\phi_1^{(i)}(r;\, \omega)\exp(\mathrm{i}\omega t)$. It is recast in terms of the outer variables, and its components are approximated as follows:

$$K_0(\sqrt{\mathrm{i}\omega}) = -\left(\gamma + \ln\sqrt{\mathrm{i}\Omega} - \Delta^{-1} + k - \ln 2\right) + \mathrm{O}(\Delta^{-1}\, \mathrm{Re}^2),$$

$$K_1(\sqrt{\mathrm{i}\omega}) = \mathrm{Re}^{-1}(\mathrm{i}\Omega)^{-1/2} + \left[\left(\gamma + \ln\sqrt{\mathrm{i}\Omega} - \Delta^{-1} + k - \ln 2\right)\left(\sqrt{\mathrm{i}\Omega}\ \mathrm{Re}\right)/2\right] + \mathrm{O}(\mathrm{Re}),$$

$$\ln r = \ln R - k + \Delta^{-1}. \tag{13}$$

Here γ is Euler's constant, while Ω is the frequency measured in terms of the outer time scale, which is therefore equal to $\omega\ \mathrm{Re}^{-2}$. The outer limit of the mode under discussion is evidently

$$\Delta\phi_1^{(i)}(r;\, \omega)\exp(\mathrm{i}\omega t) \sim \frac{\Delta}{\mathrm{Re}}\left\{\frac{1}{\sqrt{\mathrm{i}\Omega}} K_0(\sqrt{\mathrm{i}\Omega}\, R) + \frac{1}{\sqrt{\mathrm{i}\Omega}}\left(\gamma + \ln\sqrt{\mathrm{i}\Omega} + \ln R - \ln 2\right)\right\}$$

$$\times \exp(\mathrm{i}\Omega T). \tag{14}$$

It consequently gives rise to a contribution of $\mathrm{O}(\Delta/\mathrm{Re})$ to the outer expansion and this falls within the structure of solution given by relationships (5), (6) and (7).

Call the counterpart of that mode to $\psi_{2u}^{(o)}$, $\phi_2^{(o)}(R, \theta;\, \Omega)\exp(\mathrm{i}\Omega T)$. The amplitude of the vorticity associated with it is given by

$$\overline{\nabla}^2\phi_2^{(o)} = P\exp(X/2) K_0(\zeta R/2) + \chi \tag{15}$$

where

$$\zeta = \sqrt{4\mathrm{i}\Omega + 1}.$$

In this, the first term represents the amplitude of the vorticity field produced by the obstacle and prevailing close to it. Thus, this field satisfies equation (11) and its amplitude decays along any ray $\theta = \text{const}$. It also matches the vorticity in the inner field, and it is thus found that its multiplier is given by

$$P = \sqrt{\mathrm{i}\Omega + \alpha}$$

where α is arbitrary. The second term, χ, represents the amplitude of the vorticity transported from the wake into the outer field. It is obtained by integrating equation (15) and thus deriving a solution for $\phi_2^{(o)}$ which holds everywhere except along $0 \leqslant X < \infty$, $Y = 0$. It is given by

$$\phi_2^{(o)} = \sqrt{\mathrm{i}\Omega + \alpha} \int_o^\infty \exp(-\mathrm{i}\Omega\xi)\{\exp(\tilde{X}/2) K_0(\zeta\tilde{R}/2) + [\gamma + \ln(\zeta\tilde{R}/4)]\}\, \mathrm{d}\xi \tag{16}$$

where

$$\tilde{X} = X - \xi,\ \tilde{R}^2 = \tilde{X}^2 + Y^2.$$

The process of integration is very similar to that carried out in [3]. However, there the two limiting values of the second derivatives of the disturbance stream-function at $0 < X < \infty$, $Y = 0\pm$ where found to differ. In the case at hand it is the limits of the third Y-derivatives of $\phi_2^{(o)}$ that are equal in magnitude but of opposite signs. It follows that in the case at hand there is vorticity flux from the half-plane $0 < X < \infty$ into the outer field, whence the term χ. There is no such flux in the case treated in [2]. The solution (16) is in a useful form because it clearly shows that that half-plane is excluded from the outer field. An alternate form of solution is developed in the Appendix. The merit of the latter is that it provides an expression for χ and for the solution $\phi_2^{(o)}$ far downstream.

The unsteady components in the wake expansion are calculated by the standard procedure. Thus, as a first step, $\phi_2^{(o)}(X, Y, \Omega)$ is recast in terms of (X, y) as follows:

$$\frac{\Delta}{\mathrm{Re}}\phi_2^{(o)} = \frac{\Delta}{\mathrm{Re}} \sum_{n=0}^{\infty} \frac{\partial^n \phi_2^{(o)}}{\partial Y^n}(X, 0\pm)\, \mathrm{Re}^n y^n, \quad 0 < X < \infty,\ Y \gtrless 0. \tag{17}$$

This formula holds arbitrarily close to the plane $X > 0$, $Y = 0$, but not on it because the latter is excluded from the outer domain. Accordingly, the limiting values of $\phi_2^{(o)}$ or its derivatives may not be the same when that plane is approached from above and from below. But this ambiguity is immaterial because the unequal third derivatives contribute to equation (17) terms of $O(\Delta\, \mathrm{Re}^2)$. However, within the context of the present analysis, only the terms of $O(\Delta)$ in expansion (7) are accounted for.

Truncating the summation of equation (17) accordingly, one gets

$$\frac{\Delta}{\mathrm{Re}}\phi_2^{(o)} \sim \frac{\Delta}{\mathrm{Re}}\phi_2^{(o)}(X, 0) + \Delta \frac{\partial \phi_2^{(o)}}{\partial Y}(X, 0)\, y = \frac{\Delta}{\mathrm{Re}}\phi^{(w)}(X, y). \tag{18}$$

Note that both the zeroth and first derivatives are the limits obtained by approaching to plane $0 < X < \infty$, $Y = 0$ from above and from below, and that they are equal. Furthermore, this sum is not only the limit of $\phi_2^{(o)}$ at the wake. It is, in fact, the solution sought which represents the amplitude of the oscillatory disturbance in the wake flow. For, clearly, $\phi_2^{(w)} \exp(\mathrm{i}\Omega T)$ satisfies equation (1″). It is also compatible with the form (7) and matches the outer solution. Moreover, by replacing ξ by $\eta\,\mathrm{Re}^2$ and following the procedure outlined in [6] one can show that the right-hand side of relationship (18) matches $\Delta\phi_1^{(i)}$.

It has thus been shown that modes which are rotationally symmetric in the inner field are admissible. The author cannot prove that others are not but this seems to be the case. For, if one recasts $\psi_{1u}^{(i)}$ in terms of the outer variables, one finds that the components $(r^n \cos(n\theta),\ r^n \sin(n\theta))$ become $\mathrm{Re}^{-n}(R^n \cos(n\theta),\ R^n \sin(n\theta))$, just as the $\ln r$ term on the right-hand side of equation (12) contributes $\ln R$ to the right-hand side of relationship (14). However, the latter gives rise to the $\ln \tilde{R}$ term in the integrand of equation (16), which is just admissible. The terms $R^n(\cos(n\theta),\ \sin(n\theta))$ give rise to contributions that are not.

4. The form of A_0

The requirement that the disturbance flow under consideration should fall within Proudman and Pearson's scheme is evidently sufficient to determine the θ dependence of $\psi_{1u}^{(i)}$, but not its time dependence. For when $A_n(\omega)$ and $B_n(\omega)$ for $n \geqslant 1$ vanish identically, the outer limit of the inner solution has the following form:

$$\Delta\psi_{1u}^{(i)} \sim \Delta\,\mathrm{Re}\int_{-\infty}^{\infty} A_0(\Omega\,\mathrm{Re}^2)(\mathrm{i}\Omega)^{-1/2}\left[K_0(\sqrt{\mathrm{i}\Omega}R) + \left(\gamma + \ln\left(\sqrt{\mathrm{i}\Omega}R/2\right)\right)\right] \mathrm{e}^{\mathrm{i}\Omega T}\,\mathrm{d}\Omega. \tag{14'}$$

Conformity with Proudman and Pearson's scheme is attained if the non-vanishing transform A_0 is such that the Fourier integral is bounded by two constraints. It must be at most of $\mathrm{O}(\mathrm{Re}^{-2})$ as Re approaches zero. Also, as T is increased, it must remain finite. However, these two upper-bound type of constraints admit a wide spectrum of transforms A_0 representing disturbances that decay either as time progresses or as they get diffused and convected throughout the wide stream. It is well-known that perturbations in a viscous flow behave like that. Hence these are of little interest. As explained, the purpose of this piece of research is to identify unsteady disturbances that persist as time progresses and are significant over a wide portion of the flow domain.

So as to trim the form (12) further accordingly, one must identify the disturbances that are to be discarded. A typical one is a mode in which ω is complex with positive imaginary part, which represents temporal decay. Indeed such modes were not included in the form (12). An example of a disturbance one would like to discard, but which is more difficult to identify, is that represented by

$$A_0 = \left[\sqrt{\mathrm{i}\omega}\,(1+\omega^2)K_0(\sqrt{\mathrm{i}\omega})\right]^{-1}. \tag{19}$$

For with this transform the Fourier integral of relationship (14′) is $\mathrm{O}(\Delta/\mathrm{Re})$, which is vanishingly smaller than Re^{-2}. Therefore, the matching counterparts of $\psi_{1u}^{(i)}$ at the outer and wake domains are similarly small. Hence, though admissible, this disturbance is of no interest because it is significant only in the near field and has negligible traces elsewhere. This behavior is understandable noting that a measure of this disturbance's strength is given by the normalized shear at the obstacle's surface,

$$\Delta\left(\partial^2\psi_{1u}^{(i)}/\partial r^2\right)_{r=1} = \Delta\int_{-\infty}^{\infty} e^{i\omega t}(1+\omega^2)^{-1}\,d\omega = \Delta\pi\, e^{-(2\pi)|t|}, \tag{20}$$

which has a finite peak and decays with time. Thus, when transported, this perturbation becomes very thinly spread over the wide flow domain.

The last example suggests that if the disturbances sought herewith do exist, then they are represented by transforms A_0 with which the Fourier integral of relationship (14′) actually reaches the above-stated allowable upper limits. In other words, that integral is not only of $O(\mathrm{Re}^{-2})$ at most, but actually has a non-vanishing component of that order. Moreover, as T is increased, that component must not only remain finite but also non-zero. There is evidently an obvious choice of A_0 with which these requirements are satisfied. In fact the author knows of none other. It is given by

$$A_0 = \zeta(\omega)\left[(\omega-\omega_1)^{k_1}(\omega-\omega_2)^{k_2}\dots(\omega-\omega_m)^{k_m}\right]^{-1}, \tag{21}$$

were the exponents are bounded thus:

$$\sum_{j=1}^{m} k_j = 1, \tag{22}$$

while $\zeta(\omega)$ is a smooth function of ω which is finite, and integrable over $-\infty < \omega < \infty$. Here ω_j are (real) frequencies measured in terms of the inner time scale. In terms of the outer and wake time scale these are Ω_j which are equal to $\omega_j\,\mathrm{Re}^{-2}$. Therefore, when $\psi_{1u}^{(i)}$ is recast in terms of the outer variables, $\zeta(\omega)$ reduces to $\zeta(0)$, Re^{-2} can be factored out of the integral in relationship (14′) and the first of the two above-stated requirements is satisfied. Then by imposing the second, additional information concerning the exponents is gathered. For, as explained in Lighthill's [9] and other texts on Fourier transforms, their large T behavior is determined by the singularities of the integrand. In the case at hand the contributions of the j'th singularity in the right-hand side of relationship (14′) is proportional to $(T)^{k_j-1}\exp(i\Omega_j T)$. Consequently, a disturbance that is compatible with Proudman and Pearson's scheme, and also persists as T is increased and prevails in all three domains, is characterized by the form (21) in which at least one of the exponents is unity and none is bigger.

Admittedly, these considerations leave A_0 somewhat loosely defined. However, the undetermined elements in the definition of A_0 effect only the near-field flow pattern. The R and T dependence of the timewise-persistent outer limit of $\psi_{1u}^{(i)}$ due to the singularity at $\omega=\hat\omega$, is given by relationship (14) in which Ω is replaced by $\hat\Omega$, or by a sum of such terms if more than one exponent is unity. In such contribution(s) the undetermined part of A_0 plays the role of a mere factor of the form

$$\zeta(\hat\omega)\left[(\hat\omega-\omega_1)^{k_1}\dots(\hat\omega-\omega_m)^{k_m}\right]^{-1}.$$

Consequently, the outer- and wake-flow solutions representing lasting and widely-prevalent disturbances are those given by equation (16) and (18), or by a summation of expressions of that form which are characterized by distinct frequencies. The magnitude of these disturbances cannot be determined within the framework of approximation adopted here. However, their behaviour is interesting and merits a thorough exposure, which is presented in the next section.

5. Far-field behaviour

The oscillating disturbance flow due to a single mode possesses interesting features far downstream. For, when X is very large, then with a fixed value of T the streamlines' pattern representing the solution for $\phi_2^{(o)}$ is as shown in Fig. 1. It is infinitely periodic in X and moves steadily in the axial direction with the velocity of the undisturbed stream. Note that in this figure the abscissa is Y and the length scale for this variable is ν/U. The wake region is of $O(a)$ and is thus, by comparison, extremely thin. It is represented in this figure by the line $Y = 0$.

The persistence of the pattern shown in Fig. 1 is at first glance surprising. For the closed-loop streamlines constitute evidence that there exist in the flow vorticity packets of alternate signs, and that these retain their strength as they move steadily downstream. Yet

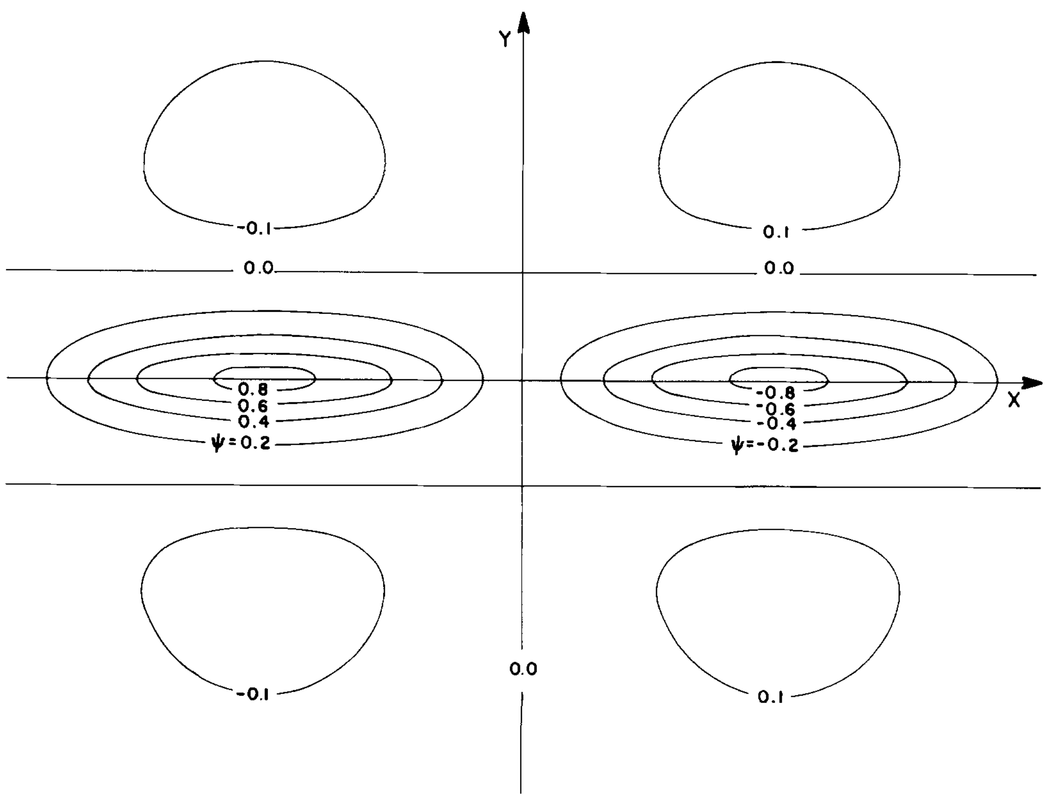

Figure 1. Oscillatory disturbance flow pattern far downstream.

it follows from the Navier-Stokes equations and the physical nature of the medium, that in the presence of viscosity vorticity decays, particularly when the Reynolds number is small and that effect is dominant. However, the point made here is that the flow pattern shown in Fig. 1 does not represent either reality or an exact solution of the Navier-Stokes equations. It is an approximation and should so be viewed. It shows that most of the vorticity generated at the obstacle's surface indeed decays, in compliance with the theory of viscous flows. The persistent pattern is induced by a residual, very small amount of vorticity, which decays at a rate that is too slow to be accounted for by the approximate solution.

The last statement shall now be mathematically substantiated. Clearly equation (15) is recovered by taking the Laplacian of relationship (16), anywhere except along the line $Y = 0, 0 < X < \infty$. The first term on the right-hand side of equation (15), which represents the amplitude of the vorticity close to the obstacle, decays and eventually vanishes as one proceeds downstream along any line $|Y| = \text{constant}$, no matter how close it is to the X axis. By elimination the residual vorticity, which produces the closed-loop streamlines, is that present in the wake which diffuses into its close vicinity in the outer field. Indeed, according to the Kaplun-Proudman and Pearson's approximation scheme, the vorticity in the wake does not decay because, unlike the governing equations for the other regions, (10) and (11), equation (1″) does not contain temporal or axial derivatives.

It remains to be shown that the residual vorticity is in the form of packets of alternate signs but equal strengths. To show this, use is made of the Fourier-integral form of solution for $\phi_2^{(o)}$, which is developed in the Appendix along the lines of Bentwich's [6] treatment of semi-bounded flow. Taking the Laplacian of (A.2) one finds that the amplitude of the vorticity associated with oscillatory disturbance-flow is given by

$$\nabla^2\phi_2^{(o)} = (\mathrm{i}\Omega + \alpha)^{1/2}\Omega(\mathrm{i}\pi/2)\int_{-\infty}^{\infty} \exp(\mathrm{i}\gamma X)\left[\gamma^2 + \mathrm{i}(\gamma + \Omega)\right]^{-1/2}$$
$$\times\exp\left(\mp\left[\gamma^2 + \mathrm{i}(\gamma+\Omega)\right]^{1/2} Y\right)\left[\delta(\gamma+\Omega) + \left(\mathrm{i}\pi(\gamma+\Omega)\right)^{-1}\right] \mathrm{d}\gamma, \quad Y \lessgtr 0, \tag{23}$$

where the branch of the square root is so chosen that its real part is positive. As in the previous section, the asymptotic behaviour of the Fourier transform is obtained by examining the singularities of the integrand. Evidently those associated with the bracketed expression raised to the power $-1/2$ represent the manner of decay of the first term on the right-hand side of equation (15). The residual portion of the vorticity, which is present in the wake and does not decay as one proceeds downstream, is due to the singularity at $\gamma = -\Omega$, and it is given by

$$\chi \exp(\mathrm{i}\Omega T) = \nabla^2\phi_2^{(o)} \exp(\mathrm{i}\Omega T)$$
$$\sim (\mathrm{i}\Omega + \alpha)^{1/2} H(X) \exp(-\mathrm{i}\Omega(T - X)) \exp(\mp|\Omega| Y),$$
$$X \to +\infty,\ Y = \text{constant} \gtrless 0. \tag{24}$$

Here H is Heaviside's unit-step function implying that there are vorticity packets for $X \to \infty$, but not upstream.

6. Possible relation to the Von Karman vortex-street

The complete flow-field downstream, namely that consisting of a single oscillatory disturbance mode superposed on a uniform stream (the steady disturbance $\psi_{2s}^{(o)}$ dies out there), is shown by Figs. 2a and 2b. The latter, depicting a relatively strong disturbance, bears a marked resemblance to the Von Karman vortex-street. It is also noted that with a single mode the inner solution $\Delta(\psi_{1s}^{(i)} + \psi_{1u}^{(i)})$ shows that the stagnation points oscillate about their mean position, as observed. This remark should not be construed as a claim

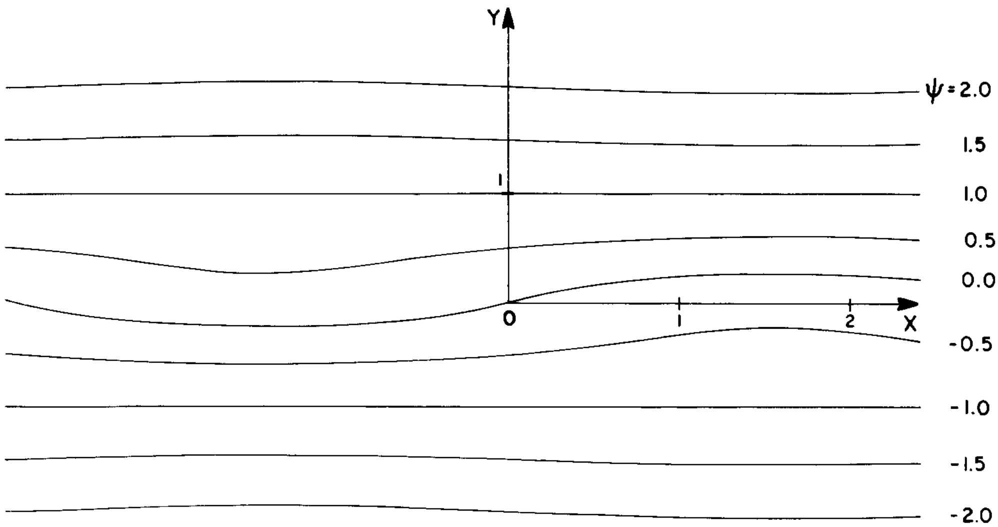

Figure 2a. Flow pattern in the wake; weak oscillatory disturbance.

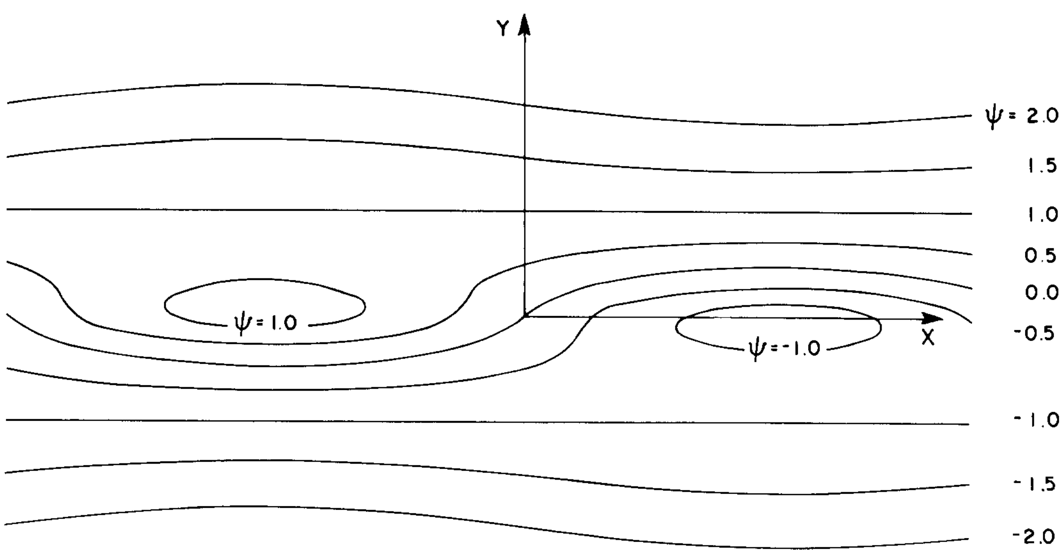

Figure 2b. Flow pattern in the wake; strong oscillatory disturbance.

that the solution proposed actually represents this phenomenon. Indeed, within the framework of this article, the questions what is $A_0(\omega)$ or that part of the transform which leaves a lasting trace, and how is it determined, are left unanswered. Therefore, one does not know whether the far-downstream oscillatory pattern is weak, strong or exists altogether. Furthermore, the proposed analysis can be expected to hold when the Reynolds number is at most unity, while that phenomenon emerges when it is about 70. Nevertheless, in discussing the possible relationship between the analysis and the phenomenon, there are two points of similarity that cannot be ignored.

Firstly, the similarity is not only between the two patterns but also between the mechanisms of the vortices' production and transport. For with a single mode of disturbance the inner-field oscillatory portion of the vorticity is

$$\nabla^2\psi_{1u}^{(i)} = \sqrt{i\omega}\, K_0(\sqrt{i\omega}\, r)\, \exp(i\omega t). \tag{25}$$

Thus, over time periods π/ω, packets of vorticity of equal strength but alternate senses are produced at the surface. In the observed phenomenon it is concentrated vortices rather than packets which are generated. But in both cases the vorticity is convected straightly downstream.

Secondly, though it has not been proved, it appears that of the solutions admitted by the approximate low-Re-analysis, only those having trailing strings of vorticity packets persist as time progresses and have traces in all three domains. The only disturbances observed in the higher-Re-flow past a cylinder are those giving rise to the phenomenon under discussion, which are very similar in their form.

These two points of similarity can be partially explained as follows. Due to the general tendency of vorticity fields to vanish, disturbances can last and widely prevail only if they have the above-described rather unique structure. Thus, as explained, the strings of vorticity packets retain their strength, due to the slow rate of decay which characterizes the wake in the low-Re-solution presented herewith. The very same selection mechanism could produce the Von Karman street, if a low-rate-of-decay wake region is also present when Re is above 70. Because of the multitude of minute random perturbations, which are present in the oncoming stream and which may be of various shapes and forms, only those typified by strings of concentrated vortices will survive.

This explanation admittedly raises many questions which shall not be addressed here, e.g., how do small perturbations become sizeable and observable? However, it is supported by the somewhat conspicuous absence of three-dimensional vortex-street-like disturbances in the flow past a sphere. This is taken as an indication that there is no wake in this flow, and indeed Proudman and Pearson's solution for that case consists of only an inner and outer expansion.

7. Determinacy and uniqueness

It follows from the analysis that all modes which are rotationally symmetric in the inner field are admissible. Therefore, there appears to be no way to determine which, if any, will emerge. Put otherwise, the Proudman and Pearson type of solution for the differential system, consisting of the Navier-Stokes equations and the stated conditions, is not unique.

Note that, although it is the allowed unsteadiness which gives rise to indeterminacy, the latter is not to within additional oscillatory terms, as the analysis carried so far suggests. For the equation governing the next term in expansion (6) reads

$$\left(\frac{\partial}{\partial T}+\frac{\partial}{\partial X}-\overline{\nabla}^2\right)\overline{\nabla}^2\psi_3^{(o)}=\frac{\partial\left(\overline{\nabla}^2\psi_2^{(o)},\ \psi_2^{(o)}\right)}{\partial(X,\ Y)}. \tag{26}$$

It therefore follows that, if $\psi_2^{(o)}$ contains modes of frequencies $\hat{\Omega}$ and $-\hat{\Omega}$, then the right-hand side of this equation will contain a steady term proportional to the product of their amplitudes $a(\hat{\Omega})$ and $a(-\hat{\Omega})$. The solution for $\psi_{3s}^{(o)}$ would therefore differ from that recorded, which is based on the premise that the unsteadiness is ruled out. It follows that in the well-known expression for the drag, the coefficient of $O(\Delta^2/\mathrm{Re})$ is determinable to within a term which is in the form of a sum of products $a(\Omega)a(-\Omega)$ over all the frequencies that typify oscillatory modes.

Since the solution proposed is valid mathematically, its apparent indeterminacy is rather disturbing when viewed from a physical standpoint. Experiments with cylindrical obstacles placed in a wide stream repeatedly produce the very same outcome, i.e. the same pattern and the very same drag. In other words nature is clearly deterministic. Therefore, if the differential system under discussion produces many outcomes, as it seems to, then it follows that it does not embody all the fluid mechanics' principles employed by nature. In other words, the considerations of continuity, stress vs. rate-of-strain relation and momentum conservation appear to admit a number of flows. To determine which will be physically realized, additional principle(s) or consideration(s) must be invoked.

Observe that when reference was made to the term indeterminacy it was cautiously qualified "apparent" because the multitude of admissible solutions is inexact. However, it seems unlikely that it is this inexactness that gives rise to non-uniqueness. The Kaplun-Proudman and Pearson scheme essentially replaces the original governing equation (1) by others. Of these it is equation (1″) that differs most from the original one and, as explained, underestimates the impact of viscosity. It follows that the exact solutions for the oscillatory modes represent disturbances that decay along the wake. But these are just as admissable as the solutions proposed.

It therefore seems that it is condition (4) that opens the door to the non-uniqueness. For all that it requires is that the disturbance created by the obstacle should be finite. This condition is therefore rather loose when compared with (2) and (3), which fix the values of ψ and its derivative in a pointwise manner on $r=1$. But, and this is important, nothing as definite as conditions (2) and (3) can be prescribed at infinity, and for two reasons. Firstly, because if the stream is unbounded then the drag is balanced by its momentum deficiency. Since both quantities are a-priori unknown, then the differential system would have become over-determined had the disturbance stream-function at infinity been forced to vanish identically, or had been otherwise prescribed in a pointwise manner. The second reason is that, when standard wind — or water — tunnel experiments are conducted, care is taken that the flow should be uniform and steady upstream. Downstream the flow is left uncontrolled. Thus, when the Reynolds number is high enough, vortex streets and other deviations from uniformity occur there. Now, outside laboratory environments, in a truly unbounded flow past an obstacle, the flow downstream is even less controlled. Consequently the looseness of condition (4) and the disturbances it allows are an inherent feature of the flow under discussion.

8. Concluding remarks

As a topic the investigation of low-Reynolds-number flows is not too popular, because such flows are rarely encountered. Therefore it may have been overlooked that these flows can be relatively easily analysed and that such analyses raise rather fundamental questions. The Von Karman street phenomenon will be addressed elsewhere. The uniqueness problem appears, at present, too complex to be tackled.

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Appendix

Following [6], the solution for the amplitude of the oscillating disturbance stream-function in the outer field is written as follows:

$$\phi_2^{(o)} = \int_{-\infty}^{\infty} \exp(\mathrm{i}\gamma X)\Big[A_{\pm}(\gamma)\exp\Big(\mp\big(\gamma^2+\mathrm{i}(\gamma+\Omega)\big)^{1/2}Y\Big) + B_{\pm}(\gamma)\exp\Big(\mp(\gamma^2)^{1/2}Y\Big)\Big]\,\mathrm{d}\gamma, \quad Y \gtrless 0. \tag{A.1}$$

The branch should be chosen such that the real parts of both square roots are positive. The four functions $A_{\pm}(\gamma)$ and $B_{\pm}(\gamma)$ are determined by matching $\Delta\phi_1^{(i)}$ with $(\Delta/\mathrm{Re})\phi_2^{(o)}$, and by imposing the requirement that both expressions for $\phi_2^{(o)}$ and the first three Y derivatives obtained thereof should be equal over $-\infty < X < 0$, $Y = 0$. As could be expected with the form (A.1), it is impossible to make $\phi_2^{(o)}$ and its derivatives continuous over $0 < X < \infty$ too.

The resulting solution is evidently

$$\phi_2^{(o)} = (\mathrm{i}\Omega+\alpha)^{1/2}\Omega(\pi/2)\int_{-\infty}^{\infty}\exp(\mathrm{i}\gamma X) \times\Big[\big(\gamma^2+\mathrm{i}(\Omega+\gamma)\big)^{-1/2}\exp\Big(\mp\big(\gamma^2+\mathrm{i}(\gamma+\Omega)\big)^{1/2}Y\Big) - (\gamma^2)^{-1/2}\exp\Big(\mp(\gamma^2)^{1/2}Y\Big)\Big](\gamma+\Omega)^{-1}\Big\{\delta(\gamma+\Omega)+\big(\mathrm{i}\pi(\gamma+\Omega)\big)^{-1}\Big\}\,\mathrm{d}\gamma. \tag{A.2}$$

Thus, replacing (X, Y) by (x, y) Re, Ω by $\omega\,\mathrm{Re}^{-2}$ and γ by β/Re, one finds that the inner limit of this component of the outer solution is

$$(\Delta/\mathrm{Re})\phi_2^{(o)} - \Delta(\mathrm{i}\omega)^{-1/2}(1/2)\int_{-\infty}^{\infty}\Big[(\beta^2+\mathrm{i}\omega)^{-1/2}\exp\big(\mp(\beta^2+\mathrm{i}\omega)^{1/2}y\big) - (\beta^2)^{1/2}\exp\big(\mp(\beta^2)^{1/2}y\big)\Big]\exp(\mathrm{i}\beta x)\,\mathrm{d}\beta, \tag{A.3}$$

and this is in agreement with equation (14). Now $\phi_2^{(o)}$ and its second Y derivative are continuous on $-\infty < X < 0$, $Y = 0$ because the right-hand side of equation (A.2) is even in Y. The continuity of $\partial\phi_2^{(o)}/\partial Y$ follows from its vanishing on $Y = 0$, $-\infty < X < 0$. The two limits of the third derivatives obtained from equation (A.1) are

$$\partial^3\phi_2^{(o)}/\partial Y^3\,|_0 = \mp(\mathrm{i}\Omega+\alpha)^{1/2}(\Omega)(\pi/2) \times\int_{-\infty}^{\infty}\exp(\mathrm{i}\gamma X)\big\{\delta(\gamma+\Omega)+(\mathrm{i}\pi(\gamma+\Omega))^{(-1)}\big\}\,\mathrm{d}\gamma, \tag{A.4}$$

and these differ in their signs. However, the expression between the curly brackets is the Fourier transform of Heaviside's unit-step function. Consequently both limits of the third derivatives vanish over $-\infty < X < 0$, $Y = 0$, as required. They are found to be of equal magnitude and opposite sign along the other half of the X axis.

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